On the Brownian motion in Lie groups*

Nikolas Koumpis†

Yale University

Contents

Abstract

In the case of a differentiable manifold, where metric properties interact with the smooth structure, Brownian motion is generated by the Laplace-Beltrami operator. On top of that, the symmetries of a Lie group can potentially facilitate implementation via the associated Lie algebra. These notes investigate the possibility of matching infinitesimal symmetries of a general smooth structure via a Lie group, with a goal of achieving more tractable implementations.

1 Introduction

The symmetries of a mathematical object form naturally the algebraic structure of a group. Symmetries often enable the development of efficient algorithms. For instance Gaussian elimination relies on the symmetries of the Euclidean space to transform an initial system of linear algebraic equations to a more simple one. Symmetry is leveraged in optimization when constrained problems recast into unconstrained ones over a certain group. Sensible functions between groups serve as a means for exposing their interaction and facilitate understanding the one by studying the other. More importantly, computational complexity does not necessarily remain unaltered under such functions, thus suggesting that it might be easier to *design* in a certain group aiming to apply in its image.

These notes focus on Lie groups. The latter are smooth manifolds equipped with a compatible group structure. The group operations (multiplication and inversion) are smooth with the corresponding isomorphisms rendering the tangent bundle of a Lie group trivial. Because of that, the operation of rather complex constructions on general manifolds, can be naturally captured

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†nikolaos.koumpis@yale.edu

by more simple ones through the associated Lie algebra. We want understand the Brownian motion on general Lie groups. In order to do so, we focus on deriving the corresponding invariant Laplace-Beltrami operator. We do so, without any compactness assumption on the group.

2 Preliminaries

A smooth manifold M of dimension $\dim(M) = d$ is a Hausdorff space along with a local diffeomorphism onto $\mathbb{R}^d$. The tangent space at a point is isomorphically defined by the space of derivators and the set of all tangent spaces admits naturally the structure of a fiber bundle$-$the tangent bundle. Then, one has a connection that indicates how changes in the total space induce changes along the fiber. Whenever a certain connection is defined, one has a notion of curvature. In what follows we review the derivation of the Laplace-Beltrami operator.

To begin with, on a smooth manifold M, consider a smooth, symmetric, covariant tensor field $g : C^\infty(TM) \times C^\infty(TM) \to C^\infty(M)$, with

$$g(X,Y)(p) = \sum_{i,j} g_{i,j}(p)\ e^i_p(X_p) \otimes e^j_p(Y_p),\ \ p \in U \tag{1}$$

where $\{e^i\}_{i=1}^d$ is the dual to the frame $\{\mathrm{E}_i\}_{i=1}^d$; the latter naturally defined by the isomorphism $\varphi_*^{-1} : T_{\phi(p)}(M) \to T_p(M),\ \ \varphi_*^{-1}(\frac{\partial}{\partial x_i}\ |_{x=\varphi(p)}) = \mathrm{E}_{i,p}$. Here (U,φ) denotes a smooth chart of the (smooth) covering of M. Provided with a smooth section $X : M \to TM$, with $X_p = \sum_{j=1} X_{i,p}\mathrm{E}_i(p)$ in $U \subseteq M$, let $X^* : M \to T^*M$ be the smooth co-vector field determined by X and the Riemannian metric g:

$$X^*(p) \equiv g(X)(Y)(p) = \sum_j X^{*j}_p e^j_p(Y), \tag{2}$$

where

$$X^{*,j}_p = \sum_i g_{i,j}(p) X_{i,p}, \tag{3}$$

are the components of X^* w.r.t. the co-frame[1] $\{e^i(\cdot)\}_i$. Conversely, given a smooth co-vector field $X^* : M \to T^*M$ as in (2), (3) determines a smooth vector field $X : M \to TM$ that in U can be expressed w.r.t. the frame $\{\mathrm{E}_i\}$ as above. Its components are

$$X_{i,p} = \sum_j g^{i,j}(p) X^{*,j}_p, \tag{4}$$

where $g^{i,j}$ represents g^{-1} in U. We isomorphically define the gradient $\mathrm{grad} f : M \to TM$ of a $0-$differential form $f \in C^\infty(M)$ to be the vector field such that

$$g(\mathrm{grad} f, Y)(p) = \mathrm{d}f(Y)(p),\ \ \forall p \in M, \tag{5}$$

where $\mathrm{d}f : \Omega^0 \to \Omega^1$ is the exterior derivative of the $0-$form f

$$\mathrm{d}f(X) \equiv Xf, \tag{6}$$

[1]The topology of the tangent bundle is given by the pre-image of the projection map. Thus, open sets of the tangent bundle are of the form $W = \pi^{-1}(U) \equiv U \times E^n$. This implies that frames and co-frames are considered always w.r.t. some open chart. Besides, they are both constant assignments only in some open $U \subseteq M$.

where of course $Xf(p) \equiv X_p f$. Inside (U, φ),

$$\mathrm{d}f(Y_p)(p) = \sum_i c^i e^i(Y_p),$$

which for $Y_p = E_{k,p}$ gives $c^k = E_{k,p} f$. This implies

$$\mathrm{grad} f(p)_i = \sum_j g^{i,j}(p) \mathrm{E}_{i,p} f,$$

or

$$\mathrm{grad}(f)(p) = \sum_i \Big(\sum_j g^{i,j}(p)(E_{j,p} f) \Big) \mathrm{E}_{i,p}, \ p \in U.$$

Take $h \in C^\infty(M)$. Then

$$\mathrm{grad} f_p(h) = \sum_i \Big(\sum_j g^{i,j} \circ \varphi^{-1}(x) \frac{\partial}{\partial x_j}\Big|_{x=\varphi(p)} \widehat{f} \Big) \frac{\partial}{\partial x_j}\Big|_{x=\varphi(p)} \widehat{h}.$$

which holds for all $\widehat{h} \in C^\infty(\mathbb{R}^{\dim(M)=d})$, and thus for $\widehat{h} = x_l$ too. As a result

$$\mathrm{grad} f_x = \sum_j g^{i,j} \circ \varphi^{-1}(x) \frac{\partial \widehat{f}}{\partial x_j}\Big|_x, x \in \bar{U} = \varphi(U) \subseteq \mathbb{R}^n,$$

Assuming that M is oriented, let $\omega_g : M \to \Lambda^n(T^*M)$ be a non-negative differential form with

$$\omega_g(X_1, \ldots, X_n)(p) = \alpha(p) \ e_p^1 \wedge \cdots \wedge e_p^n(X_1, \ldots, X_n), \ p \in (U, \varphi) \tag{7}$$

where $\alpha \in C^\infty(M)$, and $\{e^i\}_i$ is the dual to the coordinate frame $\{\mathrm{E}_i\}_i$ in (U, φ). In addition, given the metric (1), let $\{\widehat{\mathrm{E}}_i\}_i$ be the orthonormal frame w.r.t. to which we assign positive orientation in (U, φ), and $\{\hat{e}_i\}_i$ the corresponding dual. With respect to this frame, the Riemannian volume form reads

$$\omega_g(X_1, \ldots, X_n)(p) = \widehat{e}_p^1 \wedge \cdots \wedge \widehat{e}_p^n(X_1, \ldots, X_n), \tag{8}$$

Since the orthonormal frame spans point-wise the corresponding tangent space, and since $\widehat{e}_p^i = \lambda_1 e_p^1 + \lambda_2 e_p^2 + \cdots + \lambda_n e_p^n$, we obtain $\widehat{e}_p^i(\mathrm{E}_{k,p}) = \lambda_k$., $\widehat{e}_p^i(E_{k,1}) = c_k^i$, and therefore $\widehat{e}_i^i = \sum_j c_j^i e_p^j$. By equating (7), and (8) on the coordinate frame components, we obtain:

$$\begin{aligned}
\omega(\mathrm{E}_{1,p}, \ldots, \mathrm{E}_{n,p}) &= \widehat{e}_p^1 \wedge \cdots \wedge \widehat{e}_p^n (\mathrm{E}_{1,p}, \ldots, \mathrm{E}_{n,p}) \\
&= \mathrm{Alt}\left(\widehat{e}_p^1(\mathrm{E}_{1,p}) \otimes \cdots \otimes \widehat{e}_p^n(E_{n,p}) \right) \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \widehat{e}_p^1 \left(\mathrm{E}_{\sigma(1),p} \right) \otimes \cdots \otimes \widehat{e}_p^n \left(\mathrm{E}_{\sigma(n),p} \right).
\end{aligned}$$

Further,

$$\widehat{e}_p^i \left(\mathrm{E}_{\sigma(i)p} \right) = \widehat{e}_p^i \left(\varphi_*^{-1} \left(\partial x_{\sigma(i)}\Big|_{x=\varphi(p)} \right) \right) = (\varphi^{*-1} \widehat{e}_1^i) \left(\partial x_{\sigma(i)}\Big|_{x=\varphi(p)} \right)$$

$$= \left(\varphi^{*-1}\sum_j c^i_j e^j_p\right)\left(\partial x_{\sigma(i)}\Big|_{x=\varphi(p)}\right) = \left(\sum_j c^i_j \varphi^{*-1} e^j_p\right)\left(\partial x_{\sigma(i)}\Big|_{x=\varphi(p)}\right)$$

$$= \sum_j c^i_j dx^j \left(\partial x_{\sigma(i)}|_{x=\varphi(p)}\right) = c^i_{\sigma(i)}(\varphi(p)) = c^i_{\sigma(i)}(x).$$

Therefore,

$$\alpha(p) = \omega\left(\mathrm{E}_{1,p},\dots,\mathrm{E}_{n,p}\right) = \frac{1}{n!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)c^1_{\sigma(1)}\cdots c^n_{\sigma(n)} = \det\left(C^\top(\varphi(p))\right).$$

On top of that, since $\{\widehat{\mathrm{E}}_{i,p}\}_i$ are orthonormal w.r.t. the Riemannian metric g,

$$g_{i,k}(p) = g(\mathrm{E}_{i,p},\mathrm{E}_{k,p})(p) = g\Big(\sum_j c^j_i \widehat{\mathrm{E}}_{j,p}, \sum_\rho c^\rho_k \widehat{\mathrm{E}}_{\rho,p}\Big)(p) = \sum_{j,\rho} c^j_i c^p_k \delta^j_\rho = \left[CC^\top\right]_{i,k}.$$

As a result, $\alpha(p) = \sqrt{\det(g(p))}$. At this point we assume that given $U \subseteq M$, we can find an oriented orthonormal frame $\{\mathrm{E}_i\}^n_i$, with co-frame $\{e_i\}^n_i$.

Lemma 1. *On a smooth oriented Riemannian manifold M, the divergence of a smooth section $X: M \to TM$, inside a smooth chart (U,φ) is expressed as*

$$\operatorname{div}(X)(p) = \frac{1}{\sqrt{\det(g(p))}}\sum_i \mathrm{E}_{i,p}\big(\sqrt{\det(g(p))}X_i(p)\big),\ \ p\in U \tag{9}$$

where $\{E_i\}_i$ is the natural frame in (U,φ).

Proof. By definition

$$\operatorname{div}(X) \equiv \star^{-1}\mathrm{d}\big(X \lrcorner\, \omega_g\big), \tag{10}$$

where, $\star : \Omega^n(M) \to C^\infty(M)$ is the hodge star operator, $\lrcorner : \Omega^n(M) \to \Omega^{n-1}(M)$, the interior product, and $\mathrm{d} : \Omega^n(M) \to \Omega^{n+1}(M)$ the exterior derivative.

In a smooth chart (U,φ) with associated coordinate frame $\{\mathrm{E}_i\}_i$, and corresponding dual $\{\mathrm{e}_i\}_i$, the interior product can be expressed explicitly. In particular, the interior product of X and the ω_g is a reduced order alternating tensor with value at $p\in U$

$$(X \lrcorner\, \omega_g)(p) = X \lrcorner \left(\sqrt{\det(g(p))}e^1_p \wedge\cdots\wedge e^n_p\right). \tag{11}$$

All $0-$tensors are alternating tensors, and, the wedge product of a $0-$tensor and any alternating tensor is well-defined. On top of that, the interior product of a $0-$ differential form is zero and thus, by the product rule of interior product

$$(X \lrcorner\, \omega_g)(p) = (-1)^0\sqrt{\det(g(p))}\big(X \lrcorner(e^1_p\wedge\cdots\wedge e^n_p)\big)(p)$$

$$= \sqrt{\det(g(p))}\sum_{i=1}^n(-1)^{i-1}e^i_p(X)\wedge\dots\widehat{e}^i_p\cdots\wedge e^n_p$$

$$= \sqrt{\det(g(p))}\sum_{i=1}^n(-1)^{i-1}X_i(p)e^1_p\wedge\dots\widehat{e}^i_p\cdots\wedge e^n_p \tag{12}$$

By taking the exterior derivative of (12) and subsequently using the product rule for exterior derivatives [Lee, 2012, p. 365] we obtain the $n-$differential form, with value at $p \in U$

$$\mathrm{d}(X \lrcorner\, \omega_g)(p) = \mathrm{d}\Big(\sum_i \sqrt{\det(g(p))}(-1)^{i-1} X_i(p) e_p^1 \wedge \dots \widehat{e}_p^i \dots \wedge e_p^n \Big)$$

$$\overset{\text{linearity \& product rule of d}}{=} \sum_i \mathrm{d}\big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\big) \wedge e_p^1 \wedge \dots \widehat{e}_p^i \dots \wedge e_p^n$$

$$+ \sum_i \sqrt{\det(g(p))}(-1)^{i-1} X_i(p) \mathrm{d}(e_p^1 \wedge \dots \widehat{e}_p^i \dots \wedge e_p^n). \tag{13}$$

The second term is zero. For the first term, the differential $\mathrm{d}\big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\big)$ is a co-vector field with value at p

$$\mathrm{d}\big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\big)(Y)(p) = \sum_l \beta_l(p) e_p^l(Y), \;\; p \in U, \;\; Y \in T_pM \tag{14}$$

For $Y_p = \mathrm{E}_{\eta p}$ in (14) we obtain

$$\beta_\eta(p) = \mathrm{d}\big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\big)(\mathrm{E}_{\eta p})(p) \tag{15}$$

On top of that the exterior derivative of a $0-$form

$$\mathrm{d}f(Y)(p) \equiv Y_p f. \tag{16}$$

Therefore, from (14), and (15) we obtain

$$\beta_\eta(p) = \mathrm{E}_{\eta p} \sqrt{\det(g(p))}(-1)^{i-1} X_i(p). \tag{17}$$

Thus, the exterior derivative reads:

$$\mathrm{d}\big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\big)(Y)(p) = \sum_l \mathrm{E}_{lp} \Big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p) \Big) e_p^l$$

$$= (-1)^{i-1} \sum_l \mathrm{E}_{lp} \Big(\sqrt{\det(g(p))} X_i(p) \Big) e_p^l. \tag{18}$$

By plugging (18) into (13), we can easily observe that the only term that remains from the sum in (18) is the ith term. Thus,

$$\mathrm{d}(X \lrcorner\, \omega_g)(p) = \sum_i \Big((-1)^{i-1} \sum_l \mathrm{E}_{lp} \Big(\sqrt{\det(g(p))} X_i(p) \Big) e_p^l \Big) \wedge e_p^1 \wedge \dots \widehat{e}_p^i \dots \wedge e_p^n$$

$$= \sum_i \mathrm{E}_{ip} \Big(\sqrt{\det(g(p))} X_i(p) \Big) e_p^i (-1)^{i-1} \wedge (e_p^1 \wedge \dots \widehat{e}_p^i \dots \wedge e_p^n) \tag{19}$$

In the above expression, e_p^i swaps $i-1$ times to reach the i slot, and thus

$$\mathrm{d}(X \lrcorner\, \omega_g)(p) = \sum_i \mathrm{E}_{ip} \Big(\sqrt{\det(g(p))} X_i(p) \Big) e_p^1 \wedge \dots \wedge e_p^n$$

$$= \frac{\sum_i \mathrm{E}_{ip}\Big(\sqrt{\det(g(p))}X_i(p)\Big)}{\sqrt{\det(g(p))}}\omega_g(p)$$

As a result, the Riemannian divergence of the vector field X at a point $p \in U$ can be expressed by

$$\mathrm{div}_M X(p) = \frac{\sum_i \mathrm{E}_{ip}\Big(\sqrt{\det(g(p))}X_i(p)\Big)}{\sqrt{\det(g(p))}},\ p \in U.$$

□

Given an orientation in M, the Laplacian of a smooth function $f \in C^\infty(M)$ is defined as:

$$\Delta f(p) \equiv \mathrm{div}\left((\mathrm{d}f)^\sharp\right),$$

where $(\ )^\sharp : T^*M \to TM$ denotes the dual pairing induced by the metric. To begin with, it is

$$((\mathrm{d}f)^\sharp \lrcorner\, \omega_g)(p) = (\mathrm{grad} f \lrcorner\, \omega_g)_p \equiv \mathrm{grad} f_p \lrcorner\, \omega_{gp},$$

and from the product rule for the interior product, and given that it is zero for a real-valued function,

$$\begin{aligned} \mathrm{grad} f_p \lrcorner\, \omega_{gp} &= \sqrt{\det(g)(p)}\ \mathrm{grad} f_p \lrcorner\, (e_p^1 \wedge \cdots \wedge e_p^n) \\ &= \sqrt{\det(g)(p)}\ \sum_{i=1}^n (-1)^{i-1} e^i(\mathrm{grad} f_p) e_p^1 \wedge \cdots \wedge \widehat{e_p^i} \wedge \cdots \wedge e_p^n \\ &= \sum_{i=1}^n \Big[\sqrt{\det(g)(p)}(-1)^{i-1}\big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\,\big)\Big] e_p^1 \wedge \cdots \wedge \widehat{e_p^i} \wedge \cdots \wedge e_p^n \\ &\in \Lambda^{n-1}(T^*M). \end{aligned} \tag{20}$$

The linearity and product rule of the exterior derivative gives for (20):

$$\begin{aligned} \mathrm{d}(\mathrm{grad} f_p \lrcorner\, \omega_{gp}) = &\sum_{i=1}^n \mathrm{d}\Big[\sqrt{\det(g)(p)}(-1)^{i-1}\big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\,\big)\Big] \wedge e_p^1 \wedge \cdots \wedge \widehat{e_p^i} \wedge \cdots \wedge e_p^n \\ &+ \sum_{i=1}^n (-1)^0 \sqrt{\det(g)(p)}(-1)^{i-1}\big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\,\big)\mathrm{d}(e_p^1 \wedge \cdots \wedge \widehat{e_p^i} \wedge \cdots \wedge e_p^n) \end{aligned} \tag{21}$$

The second term in (21) is zero due to repeated indices. Further,

$$\begin{aligned} &\mathrm{d}\Big[\sqrt{\det(g)(p)}(-1)^{i-1}\big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\,\big)\Big] \\ &\qquad = \sum_{k=1}^n E_{k,p}\big(\sqrt{\det(g)(p)}(-1)^{i-1}\big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\,\big)\big)e_p^k. \end{aligned} \tag{22}$$

After plugging (22) into (21) the only non-zero component corresponds to the index i. Also, for $e_p^{k=i}$ to go from the 'zero position' to the $\widehat{e_p^i}$ position, i 'swaps' are needed and thus, (22) is multiplied by $(-1)^i$. Therefore,

$$\begin{aligned}\mathrm{d}(\mathrm{grad} f_p \lrcorner\, \omega_{gp}) &= -\Bigg[\sum_{i=1}^{n} \mathrm{E}_{i,p}\big(\sqrt{\det(g)(p)}\big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p} f)\,\big)\big)\Bigg] e_p^1 \wedge \cdots \wedge e_p^n \\ &= -\frac{1}{\sqrt{\det(g)(p)}}\Bigg[\sum_{i=1}^{n} \mathrm{E}_{i,p}\big(\sqrt{\det(g)(p)}\big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p} f)\,\big)\big)\Bigg]\omega_{gp},\end{aligned}$$

and subsequently,

$$\star^{-1}\mathrm{d}(\mathrm{grad} f_p \lrcorner\, \omega_{gp}) = -\frac{1}{\sqrt{\det(g)(p)}}\Bigg[\sum_{i=1}^{n} \mathrm{E}_{i,p}\big(\sqrt{\det(g)(p)}\big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p} f)\,\big)\big)\Bigg]$$

As a result the Laplacian reads,

$$\Delta_p f = -\frac{1}{\sqrt{\det(g)(p)}}\Bigg[\sum_{i=1}^{n} \mathrm{E}_{i,p}\Big(\sqrt{\det(g)(p)}\Big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p} f)\,\Big)\Big)\Bigg]\ ,\ p \in U. \tag{23}$$

We can further expand (23) as follows:

$$\begin{aligned}\Delta_p f &= -\frac{1}{\sqrt{\det(g)}}\Bigg[\sum_{i=1}^{n} \mathrm{E}_i\Big(\sqrt{\det(g)}\Big(\sum_j g^{i,j}(x)\,(\mathrm{E}_j f)\,\Big)\Big)\Bigg] \\ &= -\sum_{i,j}\frac{\mathrm{E}_i\big(\sqrt{\det(g)}g^{i,j}\big)}{\sqrt{\det(g)}}\mathrm{E}_j f - \sum_{i,j} g^{i,j}\mathrm{E}_i\mathrm{E}_j f \end{aligned}\tag{24}$$

the latter can be written as

$$\begin{aligned}\Delta_p f &= \sum_j\Big(\sum_i \frac{\mathrm{E}_{ip}\big(\sqrt{\det(g)}g^{i,j}\big)}{\sqrt{\det(g)}}\Big)\mathrm{E}_{jp} f + \sum_{i,j} g^{i,j}\mathrm{E}_{ip}\mathrm{E}_{jp} f \\ &= \mathbf{d}(x)^\top \nabla f + g^{-1}(x) : \mathrm{Hess} f,\end{aligned}\tag{25}$$

where $\big[\mathbf{d}(x)\big]_i = \mathrm{div}\big(g^{-1,i}(x_t)\big)$, where the divergence takes the ith column of the matrix g^{-1}. So each component of $\mathbf{d}$ is the divergence of the ith column of the matrix $g^{-1}(p)$. In other words, we may consider a vector field with components in U the entries of the ith column of $g^{-1}(p)$. Then, the generator of the Riemannian Brownian motion w.r.t. the metric g reads:

$$\Delta_p f = \mathrm{div}_M g^{-1}(p)^\top \nabla_p f + \frac{1}{2} g^{-1}(p) : 2\mathrm{Hess}_p f, \tag{26}$$

and from [Pavliotis, 2016, p. 66] we can identify the Riemannian Brownian motion as the following SDE

$$\mathrm{d}x_t = \mathrm{div}_M\big(g^{-1}(x_t)\big)\mathrm{d}t + \sqrt{2g^{-1}(x_t)}\mathrm{d}\omega_t, \tag{27}$$

where the divergence is applied to every column of the inverse metric. Worth noticing that the Riemannian Brownian motion, as opposed to the standard Brownian motion, incorporates a drift term. Intuitively, this term is indicated by the inhomogeneous local metric properties (i.e. volume distortion) and reflects the ununiform spread.

We can do more by writing the term inside the first sum in (24) as

$$\frac{\mathrm{E}_i(\sqrt{\det(g)}g^{i,j})}{\sqrt{\det(g)}} = \frac{\mathrm{E}_i(\sqrt{\det(g)})g^{i,j}}{\sqrt{\det(g)}} + \mathrm{E}_i g^{i,j} \tag{28}$$

At this point we can further express the first term in (28) by using the fact that the Riemannian metric automatically determines uniquely a Riemannian torsion-free connection on M [Helgason, 1979][2]. A connection is represented by the Christoffel symbols, which in this case, are determined by the Riemannian metric.

$$\mathrm{E}_k g_{ij} = \Gamma_{ikj} + \Gamma_{jki} = \sum_l g_{lj}\Gamma^l_{ik} + g_{li}\Gamma^l_{jk}, \tag{29}$$

$$\mathrm{E}_k g^{ij} = \sum_l -g^{lj}\Gamma^i_{lk} - g^{li}\Gamma^j_{lk}, \tag{30}$$

where (30) follows from differentiating the identity $g_{ij}g^{jk} = \delta^k_i$ and inserting (29). From the expression of the derivative of the metric determinant we also find

$$\begin{aligned}
\mathrm{E}_k \det(g) &= \sum_{i,j} \det(g) g^{ij} \mathrm{E}_k g_{ij} \\
&= \sum_{i,j} \det(g) g^{ij} \left(\Gamma_{ikj} + \Gamma_{jki}\right) \\
&= \sum_{i,j} \det(g)(\Gamma^i_{ik} + \Gamma^j_{jk}) = 2\sum_i \det(g)\Gamma^i_{ik}
\end{aligned}$$

From this we obtain

$$\sum_i \Gamma^i_{ik} = \frac{1}{2\det(g)} E_k \det(g) = \frac{1}{\sqrt{\det(g)}} E_k \sqrt{\det(g)}.$$

Thus, (28) reads

$$\frac{\mathrm{E}_i(\sqrt{\det(g)}g^{i,j})}{\sqrt{\det(g)}} = \sum_l \Gamma^l_{l,i} g^{i,j} + \sum_l -g^{l,j}\Gamma^i_{l,i} - \sum_l g^{l,i}\Gamma^j_{l,i},$$

and subsequently the first term in (24)

$$\sum_{i,j} \frac{\mathrm{E}_i(\sqrt{\det(g)}g^{i,j})}{\sqrt{\det(g)}} \mathrm{E}_j f = \sum_{i,j,l} (\Gamma^l_{l,i} g^{i,j} - g^{l,j}\Gamma^i_{l,i} - g^{l,i}\Gamma^j_{l,i}) \mathrm{E}_j f.$$

At this point we can use the symmetry of lower indexes of the Christofell symbols, isolate the second term, and swap the indexes l, and i, to obtain

$$\sum_{i,j} \frac{\mathrm{E}_i(\sqrt{\det(g)}g^{i,j})}{\sqrt{\det(g)}} \mathrm{E}_j f = -\sum_{i,j,l} g^{l,i}\Gamma^j_{l,i} \mathrm{E}_j f$$

[2] In general, a connetion indicates how motion in the total space, in that case in the tangent bundle, induces motion along the fibre (tangent space)- it is a vector-valued $1-$form.

As a result, the Laplace-Beltrami operator reads:

$$\Delta_p f = \sum_{i,j,l} g^{l,i}(p)\Gamma^j_{l,i}(p)\mathrm{E}_j f(p) - \sum_{i,j} g^{i,j}(p)\mathrm{E}_i\mathrm{E}_j f(p),\ p \in U \subseteq M. \tag{31}$$

Again, we can identify the Riemannian Brownian motion as

$$\mathrm{d}x_t = \left[\sum_{i,l} g^{l,i}(x_t)\Gamma^j_{l,i}(x_t)\right]_{j=1:n} \mathrm{d}t + \sqrt{2g^{-1}(x_t)}\mathrm{d}\omega_t, \tag{32}$$

where explicitely, the drift term depends on the Riemannian connection. Both stochastic differential equations (27), and (32) are understood in the Ito sense.

3 In Lie groups

Under valid collections of open sets we are in a position to model continuous symmetries of topological groups. The line, the circle, the invariances of the Euclidean space are all standard examples of topological groups. Topological groups are central objects in areas like harmonic analysis, representation theory, and differential geometry. A very important set of topological groups are Lie groups. Lie groups are smooth manifolds that are also groups with the group operations being smooth.

For any $g \in G$, the left shift by g is the map $L_g : G \to G$ defined by $L_g(h) \equiv gh$. In addition, the right shift (by g) is the map $R_g : G \to G$ defined by $R_g(h) = hg$. Both left and right shifts are diffeomorphisms. For $g, h \in G$, the composition of left shifts L_g and L_h is: $L_g \circ L_h = L_{gh}$, which implies $L_g^{-1} = L_{g^{-1}}$. The tangent map of L_g at $h \in G$ is $L_{g*} : T_hG \to T_{gh}G$, and it is composed as $L_{g*} \circ L_{h*} = L_{gh*}$ which implies $L_{g*}^{-1} = L_{g^{-1}*}$.

$$\begin{aligned} \mathrm{d}f(Y_g)(g) &= \mathrm{d}f(L_{g*} \circ L_{g*}^{-1} Y_g)(g) \\ &= (L_g^* \mathrm{d}f)(L_{g*}^{-1} Y_g)(e). \end{aligned} \tag{33}$$

Consider a symmetric, positive definite covariant $2-$tensor on T_eG, $\tau : T_eG \times T_eG \to \mathbb{R}$, and define the Riemannian metric $\bar{g} : C^\infty(TG) \times C^\infty(TG) \to C^\infty(G)$ such that

$$\bar{g}(X_g, Y_g)(g) \equiv \tau(L_{g*}^{-1} X_g, L_{g*}^{-1} Y_g),\ g \in G, X_g, Y_g \in T_gG. \tag{34}$$

It is easy to show that $\bar{g}$ is a left-invariant Riemannian metric on G. With respect to that metric, the gradient of a function $f \in C^\infty(G)$ satisfies

$$\bar{g}(\mathrm{grad} f_g, Y_g)(g) = \mathrm{d}f(Y_g)(g),\ \forall g \in G.$$

Thus, by combining (33), and (34)

$$\tau(L_{g*}^{-1}\mathrm{grad} f_g, L_{g*}^{-1} Y_g) = (L_g^* \mathrm{d}f)(L_{g*}^{-1} Y_g)(e),\ \forall Y_g \in T_gG$$

The latter expression shows that the tangent vector $L_{g*}^{-1}\mathrm{grad} f_g \in T_eG$ uniquely determines a covector $\omega(Y_e) \equiv \tau(L_{g*}^{-1}\mathrm{grad} f(g), Y_e) \in T_e^*G$. So there is a vector space isomorphism $Q : T_eG \to T_e^*G$ such that

$$(Q \circ L_{g*}^{-1}\mathrm{grad} f_g)(Y_e)(e) = (L_g^* \mathrm{d}f)(Y_e)(e)$$

for all $Y_e \in T_eG$. Point-wise definition of equality of co-vectors yields:

$$Q \circ L_{g*}^{-1}\mathrm{grad} f_g = L_g^* \mathrm{d}f,$$

or

$$\mathrm{grad} f_g = L_{g*} \circ Q^{-1} \circ L_g^*\ \mathrm{d}f.$$

3.1 The Left-invariant Laplacian

The result in (31) relies on the smooth structure of M, and it can be transferred unaltered into the Lie group setting by plugging-in the algebraic structure provided by the group. In [Liao, 2004], the line of argumentation uses [Helgason, 1979, page 105, Theorem 1.7] and essentially zeroes down a term related to the connection on the group. Instead, our derivation follows similar and perhaps more refined arguments and refers to [Tuynman, 1995] were no assumptions regarding the affine connection are made; and with two exponential maps being involved.

Lemma 2. *The left-invariant Riemannian Brownian motion on G reads*

$$\mathrm{d}g_t = L_{g_t *}\Big(-\frac{1}{2}\sum_{i,j}\gamma^j_{ij}\epsilon_i \mathrm{d}t + \sum_i \epsilon_i(\hookleftarrow)\mathrm{d}\omega_{ti}\Big), \tag{35}$$

Before giving the proof to Lemma 2, its necessary to state the following

Definition 1. *On a smooth manifold M, consider the differential operator $D : C^\infty(M) \to C^\infty(M)$. The operator D is called invariant under the diffeomorphism $\psi : M \to M$ iff $D_{\psi(p)}f = D_p f \circ \psi$.*

Proof. The main goal is to express the coordinate frame induced by the smooth chart (U, φ) w.r.t. the left-invariant vector-fields made by a basis in the tangent space at the identity of the group, and in such a way extend the Laplacian to the entire manifold. In other words, we want to perform a transformation at every tangent space T_gG at $g \in U$.

To do so, first take U in (31) to be an open set that contains the identity of the group, and subsequently choose a basis $\{\epsilon_i\}_{i=1}^n$ on T_eG. The exponential map $\exp : T_eG \to G$ is a diffeomorphism from an open neighborhood of T_eG around zero to an open neighborhood $V \subseteq U$ [Helgason, 1979, p.104, Proposition 1.6], and therefore, with $g = \exp(x_i\epsilon_i)$, we can define $\varphi(g) = (x_1(g), \ldots, x_n(g))$ such that $x_i(e) = 0$, and $x_i(g) = x_i \in \mathbb{R}$ and close to zero.

Let now $\{\mathrm{E}_i\}_i$ be the coordinate frame inside this small neighborhood V, i.e. $\mathrm{E}_i x_k = \delta_{ik}$, or equivalently $\mathrm{E}_{ig} = \varphi_*^{-1}(\partial_i|_{x=\varphi(g)})$. Note that $\{\mathrm{E}_{ie}\}_i = \{\epsilon_i\}_i$. Consider (31) w.r.t. this frame

$$\Delta_g f = \sum_{i,j,l} g^{l,i}(g)\Gamma^j_{l,i}(g)\mathrm{E}_{jg}f - \sum_{i,j} g^{i,j}(g)\mathrm{E}_{ig}\mathrm{E}_{jg}f, \;\; g \in V. \tag{36}$$

At this point, take the group homomorphism of the line (with shift symmetry) $\gamma : t \mapsto G$, which we know it reads $\gamma = \exp(tX), \;\; X \in T_eG$, and let it act on G from the right

$$\theta_\gamma(t) \equiv R_{\gamma(t)}(g). \tag{37}$$

The infinitesimal generator $\bar{X} : G \to TG \equiv G \times T_eG$ of $\theta : \gamma \times G \to G$ is a left-invariant vector field and by definition (it is the velocity vector of the orbit of $g \in G$ under the action θ_γ), its value at $g \in G$ is given by

$$\bar{X}_g f \equiv \mathrm{d}_t f(R_{\exp(tX)}(g))\big|_{t=0}, \tag{38}$$

for all $f \in C^\infty(V \subseteq G)$. Further, $\bar{X}_g$ is a tangent vector at T_gG, and thus, it can be written w.r.t. the coordinate frame-basis as

$$\bar{X}_g = \sum_j \alpha_j(g)\mathrm{E}_{jg}. \tag{39}$$

Now, we know that every tangent vector at the identity completely determines a left-invariant vector field through the tangent map of the left-shift $L_g : G \to G$. Essentially, this is because through the additional algebraic structure we can map diffeomorphically any neighborhood of the identity to every neighborhood that contains a point of the group. By applying this to the basis $\{\epsilon_i\}_i \in T_eG$, we have:

$$\bar{\epsilon}_{ig} = \sum_j \alpha_{ij}(g) \mathrm{E}_{jg}. \tag{40}$$

Essentially, we want to find the inverse of the transformation $\{\alpha_{ij}(g)\}_{ij=1}^{\dim(G)}$. The first step to do so is to plug-in (40) into (38), and subsequently to view $g \in V$ as the member of the one-parameter sub-group $g = \exp(tY),\ Y \in T_eG,$ for $t = 1$, thus obtaining

$$\sum_j \alpha_{ij}(\exp(Y)) \mathrm{E}_{j\exp(\mathrm{Y})} f = \mathrm{d}_t f\big(\exp(Y)\exp(t\epsilon_i)\big)\big|_{t=0}. \tag{41}$$

Now, we use the formula for the product of the exponential map [Helgason, 1979, page 106]

$$\exp(Y)\exp(tX) = \exp\big(Y + t\epsilon_i + \frac{t}{2}\mathrm{ad}(Y)(\epsilon_i) + \mathrm{O}(t^2)\big),$$

which after plugging it into (41) yields:

$$\sum_j \alpha_{ij}(\exp(Y)) \mathrm{E}_{j\exp(\mathrm{Y})} f = \mathrm{d}_t f\Big(\exp\big(Y + t\epsilon_i + \frac{t}{2}\mathrm{ad}(Y)(\epsilon_i) + \mathrm{O}(t^2)\big)\Big)\big|_{t=0}. \tag{42}$$

The term inside the exponential map in the right-hand side of (42) is an element $v \in (T_eG, [\ ,\ ]) \equiv \mathfrak{g}$. Observe now that since the exponential map is a diffeomorphism, $f = x_k$ in would give us the kth component of v. That is,

$$\alpha_{ik}(\exp(Y)) = \mathrm{d}_t x_k(Y + t\epsilon_i + \frac{t}{2}\mathrm{ad}(Y)(\epsilon_i) + \mathrm{O}(t^2))\big|_{t=0},$$

or

$$\alpha_{ik}(g) = \mathrm{d}_t\big(Y_k + t(\epsilon_i)_k + \frac{t}{2}\mathrm{ad}(Y)(\epsilon_i)_k + \mathrm{O}(t^2)\big)\big|_{t=0}$$

or

$$\alpha_{ik}(g) = (\epsilon_i)_k + \frac{1}{2}\mathrm{ad}(Y)(\epsilon_i)_k,\ g \in V. \tag{43}$$

Note that the dependency on g in the right hand-side of (43) 'is hidden' in Y and arises again from the fact that exp is a diffeomorphism. With $Y = \sum_i y_i\epsilon_i \in \mathfrak{g}$, and close to zero,

$$\exp(Y) = g,$$

with $\varphi(g) = \{y_i\}_i$. Therefore, $Y = \sum_i y_i(g)\epsilon_i$, and

$$\alpha_{ik}(g) = (\epsilon_i)_k + \frac{1}{2}\mathrm{ad}\Big(\sum_\eta y_\eta(g)\epsilon_\eta\Big)(\epsilon_i)_k. \tag{44}$$

Due to linearity of the bracket

$$\alpha_{ik}(g) = (\epsilon_i)_k + \frac{1}{2}\sum_\eta y_\eta(g)\mathrm{ad}(\epsilon_\eta)(\epsilon_i)_k,\ g \in V,$$

where the terms $\mathrm{ad}(\epsilon_\eta)(\epsilon_i) \in \mathfrak{g}$ can be expresses w.r.t. the basis $\{\epsilon_i\}_i$ and the structure coefficients of the group. The tangent map of the left-shift of the ith basis vector of the tangent space at the identity at $g \in V$ reads

$$\bar{\epsilon}_{ig} = \sum_k \Big((\epsilon_i)_k + \frac{1}{2}\sum_\eta x_\eta(g)\mathrm{ad}(\epsilon_\eta)(\epsilon_i)_k\Big)\mathrm{E}_{kg},$$

where x_η are the coordinates of $g \in V$. To find the inverse transformation, we start by the fact that the tangent map of the exponential map $\exp(X)_* : T_eG \to T_{\exp(X)}G$ reads [Tuynman, 1995]

$$\exp(X)_* = L_{\exp(X)*} \circ \Big(\sum_{k=0}^{+\infty}\frac{(-1)^k}{(k+1)!}\big(\mathrm{ad}(X)\big)^k\Big),$$

where

$$\big(\mathrm{ad}(X)\big)^{k+1}(Y) = \big[X, \big(\mathrm{ad}(X)\big)^k(Y)\big], \quad \mathrm{ad}(X)(Y) = [X, Y]. \tag{45}$$

The map

$$\sum_{k=0}^{+\infty}\frac{(-1)^k}{(k+1)!}\big(\mathrm{ad}(X)\big)^k = L^{-1}_{\exp(X)*} \circ \exp(X)_* : \mathfrak{g} \to \mathfrak{g}$$

is invertible with

$$\Big(\sum_{k=0}^{+\infty}\frac{(-1)^k}{(k+1)!}\big(\mathrm{ad}(X)\big)^k\Big)^{-1} = \mathrm{id}_{\mathfrak{g}} + \frac{1}{2}\,\mathrm{ad}(X) + \sum_{k=2}^{\infty}\beta_k\,\mathrm{ad}(X)^k,$$

where the third term vanishes for X close to zero. As a result

$$\mathrm{E}_{ig} = \sum_k \Big((\epsilon_i)_k - \frac{1}{2}\sum_\eta x_\eta(g)\big[\mathrm{ad}(\epsilon_\eta)(\epsilon_i)\big]_k\Big)\bar{\epsilon}_{kg}, \tag{46}$$

or, since $\{\epsilon_i\}$ are basis vectors,

$$\mathrm{E}_{ig} = \ \bar{\epsilon}_{ig} - \frac{1}{2}\sum_{k,\eta} x_\eta(g)\gamma^k_{\eta,i}\bar{\epsilon}_{kg}, \tag{47}$$

where the constants $\gamma^k_{\eta,i} \equiv \big[\mathrm{ad}(\epsilon_\eta)(\epsilon_i)\big]_k$ are the structural coefficients of G. Thus,

$$\begin{aligned}
\mathrm{E}_{jg}\mathrm{E}_{kg}f &= \big(\bar{\epsilon}_{jg} - \frac{1}{2}\sum_{\xi,\eta} x_\eta(g)\gamma^\xi_{\eta,i}\bar{\epsilon}_{\xi g}\big)\big(\bar{\epsilon}_{kg}f - \frac{1}{2}\sum_{\rho,\lambda} x_\lambda(g)\gamma^\rho_{\lambda,k}\bar{\epsilon}_{\rho g}f\big) \\
&= \bar{\epsilon}_{jg}\bar{\epsilon}_{kg}f - \frac{1}{2}\bar{\epsilon}_{jg}\big(\sum_{\rho,\lambda} x_\lambda(g)\gamma^\rho_{\lambda,k}\bar{\epsilon}_{\rho g}f\big) - \frac{1}{2}\sum_{\xi,\eta} x_\eta(g)\gamma^k_{\eta,i}\bar{\epsilon}_{\xi g}(\bar{\epsilon}_{kg}f) \\
&+ \frac{1}{4}\Big(\sum_{\xi,\eta} x_\eta(g)\gamma^k_{\eta,i}\bar{\epsilon}_{\xi g}\Big)\Big(\sum_{\rho,\lambda} x_\lambda(g)\gamma^\rho_{\lambda,k}\bar{\epsilon}_{\rho g}f\Big).
\end{aligned} \tag{48}$$

Let us now consider a symmetric, positive definite covariant $2-$tensor on the tangent space at the identity, $\tau : T_eG \times T_eG \to \mathbb{R}$, and we impose an extra condition to the basis $\{\epsilon_i\}_i$. That is, we assume that it is orthonormal w.r.t. τ, the latter being defined as

$$\tau(X, Y) \equiv \sum_{i,j} q_{i,j}e_i(X) \otimes e_j(Y), \ \forall X, Y \in T_eG, \tag{49}$$

where $\{e_i\}_i \in T_e^*G$ is the dual basis to $\{\epsilon_i\}$ In that case orthonormal means $\tau(\epsilon_{ie}, \epsilon_{ke}) = \sum_{j,\rho} q_{ij} q_{pk} \delta_{j\rho} = [QQ^\top]_{ik}$, where $Q : T_eG \to T_e^*G$ is the vector space isomorphism induced by the tensor (inner product). By choosing $Q = I$, $\tau(\epsilon_{ie}, \epsilon_{ke}) = \delta_{ik}$.

Subsequently, by utilizing τ, we can impose an extra condition on the Riemannian metric, that is, to be left-invariant. On top of that left-invariant Riemannian metrics $\bar{g} : C^\infty(TG) \times C^\infty(TG) \to C^\infty(G)$, are uniquely determined via the symmetric, positive definite covariant $2-$tensor on the tangent space at the identity, according to the prescription

$$\bar{g}(X_g, Y_g)(g) \equiv \tau(L_{g*}^{-1} X_g, L_{g*}^{-1} Y_g),\ g \in G, X_g, Y_g \in T_gG. \tag{50}$$

From (47), (50) we can express the metric components w.r.t. the left-invariant vector field as

$$\begin{aligned}
\bar{g}_{ij}(g) &= \bar{g}_{ij}\left(\mathrm{E}_{ig}, \mathrm{E}_{jg}\right)(g) \\
&= \bar{g}\left(\bar{\epsilon}_{ig} - \frac{1}{2}\sum_{k,\eta} x_\eta(g)\gamma_{\eta i}^k \bar{\epsilon}_{kg}\ ,\ \bar{\epsilon}_{jg} - \frac{1}{2}\sum_{k,\eta} x_\eta(g)\gamma_{\eta j}^k, \bar{\epsilon}_{kg}\right) \\
&= \delta_{ij} - \frac{1}{2}\sum_{\eta} x_\eta(g)\left(\sum_k \gamma_{\eta i}^k \delta_{kj}\right) - \frac{1}{2}\sum_{\eta} x_\eta(g)\left(\sum_k \gamma_{\eta j}^k \delta_{ki}\right) \\
&\quad + \frac{1}{4}\sum_{k,\sigma,l,\eta} x_\eta(g)\gamma_{\eta i}^k x_\sigma(g)\gamma_{\sigma_j}^l \delta_{kl} \\
&= \delta_{ij} - \frac{1}{2}\sum_{\eta} x_\eta(g)\gamma_{\eta i}^j - \frac{1}{2}\sum_{\eta} x_\eta(g)\gamma_{\eta j}^i + \frac{1}{4}\sum_{k,\eta,\sigma} x_\eta(g) x_\sigma(g)\gamma_{\eta,i}^k \gamma_{\sigma l}^k
\end{aligned} \tag{51}$$

In addition, the connection coefficients in (36) read:

$$\Gamma_{l,i}^j(g) = \frac{1}{2}\sum_s \bar{g}^{js}(g)(\mathrm{E}_{ig}\bar{g}_{sl} - \mathrm{E}_{sg}\bar{g}_{li} + \mathrm{E}_{lg}\bar{g}_{is}), \tag{52}$$

where

$$\begin{aligned}
E_{ig}\bar{g}_{sl} &= -\frac{1}{2}\gamma_{is}^l - \frac{1}{2}\gamma_{il}^s + O\left(x\right) \\
E_{sg}\bar{g}_{li} &= -\frac{1}{2}\gamma_{sl}^i - \frac{1}{2}\gamma_{si}^l + O\left(x\right) \\
E_{lg}\bar{g}_{is} &= -\frac{1}{2}\gamma_{li}^s - \frac{1}{2}\gamma_{ls}^i + O\left(x\right)
\end{aligned} \tag{53}$$

Unfortunately (36) needs the coefficients of the inverse Riemannian metric, and obtaining them from the elements of the metric is nothing but apparent. To this end, we are going to apply Definition 1 to the Laplacian, which alternatively, can be defined as:

$$\Delta_p f \equiv \sum_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} f\left(\mathbf{exp}_p\left(tY_i\right)\right)\Big|_{t=0}, \tag{54}$$

where $\{Y_i\} \in T_pM$ is any orthonormal basis of T_pM, and $\mathbf{exp} : T_pM \to U \ni p$ is the Riemannian exponential map. According to (54), the Laplace operator measures the local curvature of the real-valued map $f : M \to \mathbb{R}$ at the point $p \in M$ w.r.t. M. It does so by evaluating the function f along geodesics that initiate from the point $p \in M$ and extend along Y_i. By definition, the Laplacian is invariant under coordinate changes between orthonormal bases. Based on the above characterization, it is easy to see that the Laplacian comutes with local isometries [Gallier, 2013, Proposition 16.2]:

$$\begin{aligned}\Delta_{\varphi(p)} f &= \sum_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\mathbf{exp}_{\varphi(p)}(t\varphi_* Y_i))\big|_{t=0} \\ &= \sum_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\varphi \circ \mathbf{exp}_p(tY_i))\big|_{t=0} \\ &= \Delta_p(f\circ\varphi), \end{aligned} \tag{55}$$

On top of that, it is easy to show that for a Riemannian Lie group, with a left-invariang Riemannian metric, $\varphi = L_g$ is an isometry, and

$$\Delta_g f = \Delta_e(f\circ L_g).$$

That is, to derive the expression for the left-invariant Laplacian at a point $g \in G$, all we need is its value at the identity of the group. For $g = e$ in (48), (51), and (53) we obtain

$$\mathrm{E}_{je}\mathrm{E}_{ke} f = \bar{\epsilon}_{je}\bar{\epsilon}_{ke} f + 0,$$

$$\bar{g}^{i,j}(e) = \delta_{ij},$$

$$\begin{aligned}\Gamma^j_{l_1 i}(e) &= \frac{1}{2}\sum_s \delta_{js}\left(-\frac{1}{2}\gamma^l_{is} - \frac{1}{2}\gamma^s_{ie} + \frac{1}{2}\gamma^i_{si} + \frac{1}{2}\gamma^l_{si} - \frac{1}{2}\gamma^s_{li} - \frac{1}{2}\gamma^i_{ls}\right) \\ &= \frac{1}{2}\left(-\frac{1}{2}\gamma^l_{ij} - \frac{1}{2}\gamma^j_{il} + \frac{1}{2}\gamma^i_{jl} + \frac{1}{2}\gamma^l_{ji} - \frac{1}{2}\gamma^j_{li} - \frac{1}{2}\gamma^i_{li}\right) \\ &= \frac{1}{2}\left(\gamma^l_{ji} + \gamma^i_{jl}\right),\end{aligned}$$

respectively. Therefore,

$$\Delta_e = -\sum_i \bar{\epsilon}_{ie}\bar{\epsilon}_{ie} + \sum_{i,j} \gamma^j_{ij}\bar{\epsilon}_{ie}.$$

As a result, due to left-invariance, the left-invariant Laplacian on G reads:

$$\Delta_g = -\sum_i \bar{\epsilon}_{ig}\bar{\epsilon}_{ig} + \sum_{i,j}[\mathrm{ad}(\epsilon_i)(\epsilon_j)]_j \bar{\epsilon}_{ig}. \tag{56}$$

Note that (36) corresponds to the Riemannian Brownian motion on G which is given in local coordinates by either (27), or (32). On the contrary, (56) corresponds to the *left-invariant* Riemannian Brownian motion on G

$$\mathrm{d}g_t = L_{g_t *}\Big(-\frac{1}{2}\sum_{i,j}\gamma^j_{ij}\epsilon_i \mathrm{d}t + \sum_i \epsilon_i(\hookleftarrow)\mathrm{d}\omega_{ti}\Big), \tag{57}$$

where $\epsilon_i(\hookleftarrow)\mathrm{d}\omega_{ti}$ denotes the injection of the individual one-dimensional *standard* Brownian motion $\mathrm{d}\omega_{ti}$, into the corresponding basis vector ϵ_i of $\mathfrak{g}$. □

Note that (35) is based on the differentiable, Riemannian, *and algebraic* structure of the manifold, while (32) only on the differentiable and Riemannian structure.

3.2 The special orthogonal group

The set

$$M = \{X \in \mathbb{R}^{n\times n} \ : \ \det(X) \neq 0\}$$

is open as the complement of the closed set $\{X \in \mathbb{R}^{n\times n} \ : \ \det(X) = 0\}$. On top of that, continuity of det induces the standard topology from $\mathbb{R}$ to M. To see that we just deploy the standard continuity argument: Take $A \in M$ and the open interval $I \equiv |\det(A)| < \epsilon$, $\epsilon > 0$. Then define $U \equiv \det^{-1}(\{|\det(A)| < \epsilon\})$ (which is open). Due to continuity of det, there exists $\delta{>}0$ s.t. the image of $||X - A||_2 < \delta$ is in I: Every invertible matrix is closely sourrounded by invertible matrices.

Together with the open set M, and the empty set, the collection of the previously constructed open sets form a valid topology T, and thus a valid topological space (M, T). On top of that, the pair $(M, \cdot)$, with the second entry referring to the standard matrix multiplication, is a group since for $X_1, X_2 \in M$, $\det(X_1 \cdot X_2) = \det(X_1)\det(X_2) \neq 0$ (the identity matrix serves as the identity of the group, and every element has a well-defined inverse).

Clearly, the map $\mathrm{L}_{\bar{X}} : M \to M$ defined as $\mathrm{L}_{\bar{X}} \equiv \bar{X} \cdot X$ is smooth, and the inverse is also smooth since it can be written as $X^{-1} = \frac{1}{\det(X)}\mathrm{adj}(X)$. Thus, $(M, T, \cdot)$ is a topological group. In fact, it is the Lie group called *the general linear group*, denoted as $\mathbb{GL}(\mathbb{R}, n)$

Moving forward, the set $F = \{\mathrm{R} \in \mathbb{R}^{n\times n} : \mathrm{R}^\top\mathrm{R} = I, \ \det(\mathrm{R}) = +1\}$ is a subset of M, and the pair $(F, \cdot)$ is a group (again $\cdot$ refers to the standard matrix multiplication). Therefore, the object $(F, T, \cdot)$ is a (sub) Lie group (of $\mathbb{GL}(\mathbb{R}, n)$) called *the special orthogonal group*, denoted as $\mathbb{SO}(\mathbb{R}, n)$.

The tangent space T_eG of every Lie group G at the identity $e \in G$ is a vector space and it can always be equipped with an additional binary operator $\star$ so that the triple $(T_eG, +, \cdot, \star)$ is a Lie algebra. Clearly the tangent space at the identity of $\mathbb{GL}(\mathbb{R}, n)$ is $\mathbb{R}^{n\times n}$, and thus $\dim(\mathbb{GL}(\mathbb{R}, n)) = n^2$. To find the tangent space of the special orthogonal group, consider a smooth group homomorphism $\gamma : (\mathbb{R}, +) \to \mathbb{SO}(\mathbb{R}, n)$. We know that the isomorphism $\gamma_* : T_0\mathbb{R} \to T_I\mathbb{SO}(\mathbb{R}, n)$, and therefore, $\omega \equiv \dot{\gamma}(0) \in T_I\mathbb{SO}(\mathbb{R}, n)$. Clearly, $\gamma(t)\gamma(t)^\top = I$ for all $t \in \mathbb{R}$, and thus

$$\dot{\gamma}(0)\gamma(0)^\top + \gamma(0)\dot{\gamma}(0)^\top = 0,$$

or

$$\omega + \omega^\top = 0.$$

That is, $T_I\mathbb{SO}(\mathbb{R}, n) = \{\omega \in \mathbb{R}^{n\times n} \ : \ \omega + \omega^\top = 0\}$, and as a result $\dim(\mathbb{SO}(\mathbb{R}, n)) = \frac{n(n-1)}{2}$. Surprisingly, $\dim(\mathbb{SO}(\mathbb{R}, 3)) = 3$. It can be shown that $(T_I\mathbb{SO}(\mathbb{R}, n), +, \cdot, [\ ,\ ])$ is the Lie algebra denoted by $\mathfrak{so}(\mathbb{R}, n)$, where $[\ ,\ ] : T_I\mathbb{SO}(\mathbb{R}, n) \times T_I\mathbb{SO}(\mathbb{R}, n) \to T_I\mathbb{SO}(\mathbb{R}, n)$. In particular, for $n = 3$, the commutator is the known cross product between vectors (a lot of intuition can be gained from here).

More importantly, the tangent space at the identity, inherits the inner product from $\mathbb{R}^{n\times n}$.

$$\tau(X_e, Y_e) \equiv \mathrm{tr}(X_eY_e^\top),$$

where $e = I$. Left-invariant Riemannian metrics are in 1-1 correspondence with the inner products in the Lie algebra. Define

$$\bar{g}(X_\mathrm{R}, Y_\mathrm{R})(\mathrm{R}) \equiv \tau(\mathrm{R}^\top X_\mathrm{R}, \mathrm{R}^\top Y_\mathrm{R}).$$

Then the gradient $\text{grad}_{\text{R}} f : \mathbb{SO}(\mathbb{R}, n) \to T\mathbb{SO}(\mathbb{R}, n)$ w.r.t. that left-invariant metric is such that

$$\bar{g}(\text{grad}_{\text{R}} f, Y_{\text{R}})(\text{R}) = \text{d}f(Y_{\text{R}})(\text{R}).$$

In the following example, we compute the gradient of a real-valued map defined on the special orthogonal group

Example 1 (Brockett [1989]). *Consider the real-valued map $f : \mathbb{SO}(\mathbb{R}, n) \to \mathbb{R}$, defined as $f(x) = \text{tr}\left(x^\top AxN\right)$, where A is symmetric, and N diagonal. It is*

$$\begin{aligned}
\text{d}f(Y_{\text{R}}) &\equiv Y_{\text{R}} f \\
&= Y_I(f \circ L_{\text{R}}) \\
&= \frac{\text{d}}{\text{d}t}\left[f(\text{R} \cdot x)\Big|_{x=\Theta(t,I)} \right]\Big|_{t=0} \\
&= \frac{\text{d}}{\text{d}t}\left[f(\text{R} \cdot I \cdot \gamma(t)) \right]\Big|_{t=0} \\
&= \frac{\text{d}}{\text{d}t} f(\text{R} \cdot \gamma(t))\Big|_{t=0} \\
&= \frac{\text{d}}{\text{d}t} f(\text{R} \cdot \exp(tY_I))\Big|_{t=0},
\end{aligned}$$

where $\exp(\cdot) = e^{(\cdot)}$*, and*

$$\begin{aligned}
\frac{\text{d}}{\text{d}t} f(\text{R} \cdot \exp(tY_e))\Big|_{t=0} &= \text{tr}\big(- Y_I \text{R}^\top A \text{R} N\big) + \text{tr}\big(Y_I N \text{R}^\top A \text{R}\big) \\
&= \text{tr}\big(Y_I(\text{R}^\top A^\top \text{R} N^\top - N \text{R}^\top A \text{R})\big) \\
&= \text{tr}\big(\text{R}^\top Y_{\text{R}}(\text{R}^\top A^\top \text{R} N^\top - N \text{R}^\top A \text{R})\big) \\
&= \text{tr}\big(Y_{\text{R}}[\text{R}(\text{R}^\top A^\top \text{R} N^\top - N \text{R}^\top A \text{R})]^\top\big) \\
&= \text{tr}\big(\text{R}^\top Y_{\text{R}}[\text{R}^\top \text{R}(\text{R}^\top A^\top \text{R} N^\top - N \text{R}^\top A \text{R})]^\top\big) \\
&= \tau(\text{R}^\top \text{R}(\text{R}^\top A \text{R} N - N \text{R}^\top A \text{R}), \text{R}^\top Y_{\text{R}}).
\end{aligned}$$

Due to uniqueness of the gradient

$$\text{grad}_{\text{R}} f = \text{R}\big(\text{R}^\top A \text{R} N - N \text{R}^\top A \text{R}\big).$$

Further,

$$\omega_*^\top = \big(\text{R}^\top A \text{R} N - N \text{R}^\top A \text{R}\big)^\top = -\big(\text{R}^\top A \text{R} N - N \text{R}^\top A \text{R}\big) = -\omega_*$$

The gradient flow reads

$$\dot{\text{R}} = -\text{R}(\text{R}^\top A \text{R} N - N \text{R}^\top A \text{R}), \tag{58}$$

and it is the left-translated version of $\omega_* \in \mathfrak{so}(3)$. As we already mentioned, for each $t \in \mathbb{R}$, $-\omega_*(t) \in \mathfrak{so}(3)$, and thus the unique one-parameter sub-group $\gamma(\tau) \equiv e^{-\tau \omega_*(t)}$, $\tau \in \mathbb{R}$, such that

$\dot{\gamma}(0) = -\omega_*(t)$. Thus, for $\tau = t + h$, we obtain $\gamma(t+h) = e^{-(t+h)\omega_*(t)} = \gamma(t)e^{-h\omega_*(t)}$, and the corresponding to (58) gradient descent reads:

$$\gamma(t+h) = \gamma(t)e^{-h\omega_*(t)},$$

or

$$\mathrm{R}(t+h) = \mathrm{R}(t)e^{-h\omega_*(t)},$$

where $\omega_*(t) = \mathrm{R}^\top(t)A\mathrm{R}(t)N - N\mathrm{R}^\top(t)A\mathrm{R}(t)$.

4 Conclusion

In a differentiable manifold, Brownian motion is generated by the Laplace-Beltrami operator. In addition, the algebraic structure of the group provides extra isomorphisms that facilitate implementation via the associated Lie algebra. These notes represent an initial step towards examining the matching of infinitesimal symmetries of a general smooth structure through a Lie group, aiming for a more manageable implementations.

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